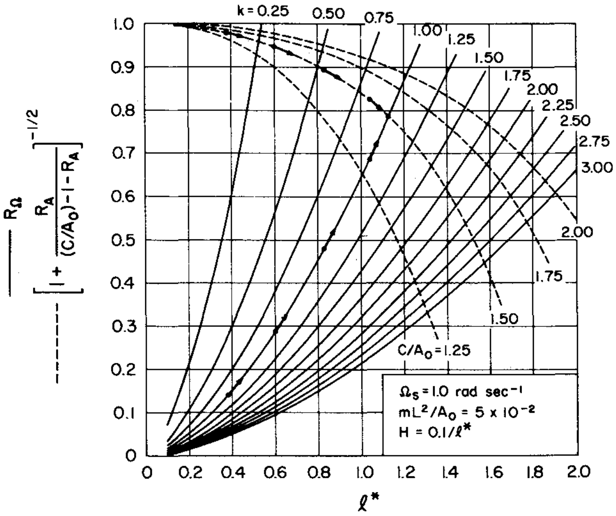

Fig. 2 Stability diagram in the parameter plane.

(25a) and (25b) are recognized as the requirement that the spin axis be that of "the greatest moment of inertia." On the other hand, the desired conditions, namely, those yielding the maximum length to which the rods can be extended without impairing stability, can be obtained from inequalities (25c) and (25d).

Numerical Results

Inequalities (25c) and (25d) can be used to derive stability diagrams in terms of the system parameters. Because information that can be extracted from inequality (25d) is analogous to that obtained from inequality (25c), we shall be concerned only with the latter.

Let us assume for simplicity that the rods are uniform, $\rho = \text{const}$, $EI = \text{const}$, and introduce the dimensionless quantities $\rho l/m = l^*$ and $h/l = H$, as well as the reference length $L = m/\rho$, which enables us to write the inertia ratio in the form

$$R_A = \frac{2}{A_0}\left[\int_h^{h+l} \rho z^2\,dz + m(h+l)^2\right] = \frac{2mL^2}{A_0}\, l^{*2}\left[1+\tfrac{1}{3}l^*(1+3H+3H^2)\right] \qquad (26)$$

The first natural frequency Λ_1 is obtained from the solution of the eigenvalue problem associated with a uniform cantilever bar with a tip mass. It can be shown to have the value

$$\Lambda_1 = k\,\frac{(\alpha_1 l)^2}{l^{*2}}, \qquad k = (EI/\rho L^4)^{1/2} \qquad (27)$$

where $\alpha_1 l$ is the lowest solution of the characteristic equation

$$(1+\cos\alpha l\cosh\alpha l) = \alpha l\,\frac{1}{l^*}(\sin\alpha l\cosh\alpha l - \cos\alpha l\sinh\alpha l) \qquad (28)$$

Corresponding to a given l^*, Eq. (28) can be solved for $\alpha_1 l$. Then for a given value of k, Eq. (27) yields Λ_1, which can be used to calculate $R_\Omega = \Omega_s/\Lambda_1$. Figure 2 shows in solid lines plots R_Ω vs l^* for $\Omega_s = 1.0$ rad/sec, with k playing the role of a parameter, and in dashed lines plots $\{1+R_A/[(C/A_0)-1-R_A]\}^{-1/2}$ vs l^* for $mL^2/A_0 = 5\times10^{-2}$ and $H = 0.1/l^*$, with C/A_0 playing the role of a parameter. Similar diagrams can be obtained for various other combinations of Ω_s, mL^2/A_0, and H. The system is stable for values of l^* for which the curve R_Ω vs l^* remains below the curve $\{1+R_A/[(C/A_0)-1-R_A]\}^{-1/2}$ vs l^*. As the antennas are being deployed slowly, hence, as l^* increases, R_Ω and $\{1+R_A/[(C/A_0)-1-R_A]\}^{-1/2}$ follow the corresponding curves, as indicated by arrows in Fig. 2. At the intersection of the two curves we have $R_\Omega = \{1+R_A/[(C/A_0)-1-R_A]\}^{-1/2}$, and beyond that point $R_\Omega > \{1+R_A/[(C/A_0)-1-R_A]\}^{-1/2}$. It follows that stability can be expected as long as l^* stays below the value corresponding to the intersection of the two curves.

As an illustration, let us consider:

$$m = 5 \text{ slug}, \quad \rho = 0.1 \text{ slug/in.}, \quad EI = 31.25\times10^6 \text{ lb in.}^2,$$
$$h = 5 \text{ in.}, \quad A_0 = 25\times10^6 \text{ slug in.}^2, \quad C = 37.5\times10^6 \text{ slug in.}^2,$$
$$\Omega_s = 1.0 \text{ rad/sec}$$

Using the above data, we can calculate the following:

$$L = m/\rho = 50 \text{ in.}, \quad k = (EI/\rho L^4)^{1/2} = 1.0, \quad C/A_0 = 1.5$$
$$mL^2/A_0 = 5\times10^{-2}, \quad H = h/Ll^* = 0.1/l^*$$

so that Fig. 2 is applicable. (This is no coincidence, as the data was purposely chosen to render Fig. 2 applicable.) The curves corresponding to $k = 1.0$ and $C/A_0 = 1.5$ intersect at $l^* = 1.13$. Hence, the motion is stable as long as $l = l^*L < 56.5$ in.

Conclusions

A method is presented whereby it is possible to determine how far a pair of antennas can be deployed without destabilizing a spinning satellite. The analysis is based on Liapunov's direct method in conjunction with the method of the integral coordinates. The numerical results are presented in the form of a parameter plot that can be used for design purposes.

References

[1] Meirovitch, L., *Methods of Analytical Dynamics*, McGraw-Hill, New York, 1970, pp. 228, 232.

[2] Meirovitch, L. and Nelson, H. D., "On the High-Spin Motion of a Satellite Containing Elastic Parts," *Journal of Spacecraft and Rockets*, Vol. 3, No. 11, Nov. 1966, pp. 1597–1602.

[3] Meirovitch, L. and Calico, R. A., "The Stability of Motion of Force-Free Spinning Satellites with Flexible Appendages," *Journal of Spacecraft and Rockets*, Vol. 9, No. 4, April 1972, pp. 237–245.

Optimal Guidance with Maneuvering Targets

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Introduction

IN the derivation of optimal guidance laws for nonlinear missile systems several inherent assumptions are made in order to allow a feedback law to be obtained. Two of the more common simplifying assumptions made are that the missile has infinite bandwidth and that the target is assumed nonmaneuvering. Cottrell,[1,2] considers an optimal derivation based upon the assumption that the missile has finite bandwidth, i.e., a first-order lag model. His optimal derivation, however, did not consider a maneuvering target but rather adds a term in the guidance law based upon intercept kinematics.

The present Note considers the development of the optimal guidance law for a missile system with finite bandwidth and with a maneuvering target. It is shown that the results reduce to

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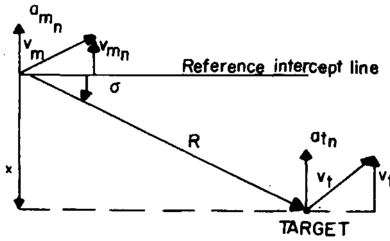

Fig. 1 Intercept kinematics.

that of Ref. 1 when the target is undergoing a constant acceleration maneuver. In order to develop the guidance laws the necessary optimal control theory is derived for a general problem and then applied to the specific problem.

General Problem and Solution

The most general problem being considered here is to minimize the quadratic performance index

$$J = \frac{1}{2}\int_{t_o}^{t_f} (X^T Q X + u^T R u)\, dt \tag{1}$$

subject to the linear nonhomogeneous differential equation constraint

$$\dot{X}(t) = A(t)X(t) + B(t)u(t) + C(t) \tag{2}$$

and zero terminal miss distance boundary constraint

$$TX(t_f) = 0 \tag{3}$$

where T is a $p \times n$ matrix and it is assumed the initial state $X(t_o)$ is given. The solution is obtained as follows:

$$J = v^T TX(t_f) + \frac{1}{2}\int_{t_o}^{t_f} (X^T Q X + u^T R u)\, dt \tag{4}$$

with v being a Lagrange multiplier vector. The necessary conditions for solution are

$$\dot{X} = AX + Bu + C \tag{5}$$

$$\dot{\lambda} = -\partial H^T/\partial X \tag{6}$$

where

$$H = \tfrac{1}{2}(X^T Q X + u^T R u) + \lambda^T (AX + Bu + C) \tag{7}$$

This gives rise to the optimal solution

$$u = -R^{-1} B^T \lambda \tag{8}$$

Now upon substitution of Eq. (8) into Eqs. (5) and (6)

$$\begin{bmatrix} \dot{X} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X \\ \lambda \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} \tag{9}$$

with $X(t_o) = X_o$ and $\lambda(t_f) = T^T v$.

One may assume a solution for Eq. (6) as

$$\lambda(t) = P(t)X(t) + S(t)v + K(t) \tag{10}$$

then

$$\dot{\lambda} = \dot{P}X + P\dot{X} + \dot{S}v + \dot{K} \tag{11}$$

The use of Eqs. (10) and (9) in Eq. (11) gives rise to the equations

$$\dot{P} + PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{12}$$

$$\dot{S} + (A^T - PBR^{-1}B^T)S = 0 \tag{13}$$

$$\dot{K} + (A^T - PBR^{-1}B^T)K + PC = 0 \tag{14}$$

The boundary conditions are found from

$$\lambda(t_f) = P(t_f)X(t_f) + S(t_f)v + K(t_f) \triangleq T^T(t_f)v \tag{15}$$

$$P(t_f) = K(t_f) = 0, \qquad S(t_f) = T^T(t_f) \tag{16}$$

Now, for the special case considered in the paper by Cottrell, $Q = 0$, and it can be verified that the solution for both P and K is zero for all time within $[t_o, t_f]$. Thus, the control law becomes

$$u(t) = -R^{-1}(t)B^T(t)S(t)v \tag{17}$$

The solution for v is obtained next. The terminal manifold may be written as functions of the initial state condition, and the Lagrange multipliers v, λ, at t_o. Thus

$$\psi(t) = U(t_o)X(t_o) + W(t_o)v + M(t_o) \tag{18}$$

or, in general

$$\psi(t) = U(t)X(t) + W(t)v + M(t) \tag{19}$$

Differentiating and equating coefficients yields

$$\begin{aligned} \dot{U} &= -U(A - BR^{-1}B^T P) \\ \dot{W} &= UBR^{-1}B^T S \\ \dot{M} &= UBR^{-1}B^T K - UC \end{aligned} \tag{20}$$

with boundary conditions

$$0 = \psi(t_f) = T(t_f)X(t_f) = U(t_f)X(t_f) + W(t_f)v + M(t_f)$$

or

$$U(t_f) = T(t_f), \qquad W(t_f) = M(t_f) = 0 \tag{21}$$

and

$$v = W^{-1}(t)\{U(t)X(t) + M(t)\} \tag{22}$$

Thus, Eq. (17) becomes

$$u(t) = R^{-1}(t)B^T(t)S(t)W^{-1}(t)\{U(t)X(t) + M(t)\}$$

with

$$\begin{aligned} &\dot{U} = -UA; \quad U(t_f) = T(t_f) \\ &\dot{W} = UBR^{-1}B^T S; \quad W(t_f) = 0 \\ &\dot{M} = -UC; \quad M(t_f) = 0 \\ &U = S^T \\ &P = K = 0 \ \forall t \varepsilon [t_o, t_f] \end{aligned} \tag{23}$$

Guidance Law Solution

The intercept geometry is shown in Fig. 1 where the following variables are defined: σ is the line of sight angle; x is the relative position; a_{T_n} is the target normal acceleration; v_{T_n} is the target normal velocity; a_{m_n} is the missile normal acceleration; v_{m_n} is the missile normal velocity; and R is the relative range vector.

The dynamics are the same as used by Cottrell.

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= a_{m_n} - a_{T_n} \\ \dot{a}_{m_n} &= -(1/\tau)a_{m_n} + (1/\tau)u \end{aligned} \tag{24}$$

Equation (2) thus becomes

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{v} \\ \dot{a}_{m_n} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1/\tau \end{bmatrix} \begin{bmatrix} x \\ v \\ a_{m_n} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/\tau \end{bmatrix} u + \begin{bmatrix} 0 \\ -a_{T_n} \\ 0 \end{bmatrix} \tag{25}$$

and the terminal constraint is

$$T(t_f)X(t_f) = 0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t_f) \\ v(t_f) \\ a_{m_n}(t_f) \end{bmatrix} \tag{26}$$

with the solution from Eq. (17) being

$$u(t) = -B^T S v \tag{27}$$

From the differential equation for S

$$\dot{S} = -A^T S; \qquad S(t_f) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{28}$$

The solution is given by

$$S(t) = \Phi(t, t_f)S(t_f) \tag{29}$$

where

$$\Phi(t, t_f) = \begin{bmatrix} 1 & 0 & 0 \\ -(t - t_f) & 1 & 0 \\ \tau^2[e^{(t-t_f)/\tau} - 1 - (t - t_f)/\tau & -\tau[e^{(t-t_f)/\tau} - 1] & e^{(t-t_f)/\tau} \end{bmatrix} \tag{30}$$

From the differential equation for W

$$\dot{W} = S^T B B^T S; \qquad W(t_f) = 0 \tag{31}$$

The solution is

$$w_{ij}(t) = 0 \,\forall\, t; \qquad i,j \neq 1 \tag{32}$$

$$w_{11}(t) = -\tau^2 \Big\{ \tau/2 - \Big[\tau/2e^{\frac{2(t-t_f)}{\tau}} - 2\tau e^{\frac{(t-t_f)}{\tau}} - 2\tau e^{\frac{(t-t_f)}{\tau}} \Big(\frac{(t-t_f)}{\tau} - 1\Big) + (1/\tau)(t-t_f)^2 + (1/2\tau^2)(t-t_f)^3 + (t-t_f) \Big]\Big\} \tag{33}$$

And finally from the differential equation for M

$$\dot{M} = -S^T C; \quad M(t_f) = 0 \tag{34}$$

One obtains the solution

$$m_2(t) = m_3(t) = 0 \tag{35}$$

$$m_1(t) = -\int_t^{t_f} -(\xi - t_f) a_{T_n}(\xi)\, d\xi \tag{36}$$

With $W(t)$ being singular, use of the term-by-term expansion of Eq. (21) rather than using Eq. (22) yields the solution for v as

$$v_2 = v_3 = 0 \tag{37}$$

$$v_1 = -\frac{1}{w_{11}(t)} \{ x - \beta v + \tau^2 (e^{\beta/\tau} - 1 - \beta/\tau) a_{m_n} + m_1 \} \tag{38}$$

Where

$$\beta \triangleq (t - t_f)$$

The control, from Eq. (27), is then

$$u(t) = \frac{\Lambda}{(t_g)^2} \{ x + t_g v + \tau^2 (e^{-T} - 1 + T) a_{m_n} + m_1 \} \tag{39}$$

where

$$t_g \triangleq -\beta = (t_f - t)$$

$$T \triangleq t_g/\tau$$

$$\Lambda \triangleq \frac{(e^{-T} - 1 + T)}{\Big\{ \frac{1}{2T^2} e^{-2T} + \frac{2}{T} e^{-T} - \frac{T}{3} - \frac{1}{T} - \frac{1}{2T^2} + 1 \Big\}} \tag{40}$$

Comparison of Results

By comparing the above guidance law with that derived by Cottrell for the nonmaneuvering target it is seen that the difference is an additional term $(\Lambda m_1/t_g^2)$ for the present guidance law. If the target is undergoing an evasive maneuver of a constant g turn, then the target acceleration is constant in Eq. (36) and then Eq. (39) becomes

$$u(t) = \frac{\Lambda}{(t_g)^2} \Big\{ x + t_g v + \tau^2 (e^{-T} - 1 + T) a_{m_n} - \frac{(t_g)^2}{2} a_{T_n} \Big\} \tag{41}$$

This agrees with the intuitive derivation of Cottrell.

Possible Applications

In a general problem one may not have knowledge of future target acceleration. However, in the problem of a missile intercepting another missile, called the target missile, which in turn is homing on a bomber, the target missile undergoes a sinusoidal acceleration profile due to its response to the scintillation noise phenomenon. Thus, the frequency and amplitude of the acceleration may be obtained by system analysis. The phase information may be obtained by measurements of the missile velocity. In a problem of an anti-ballistic missile intercepting a re-entry vehicle (RV) one has knowledge of the flight path of the RV and, thus, its acceleration profile. Similarly, in an anti-satellite missile the target acceleration is a known quantity. Thus, the result presented here which requires knowledge of future target acceleration may be quite useful in several scenarios.

Summary and Conclusions

This Note presents the analytic derivation of the optimal closed-loop guidance law for a finite-bandwidth missile intercepting a maneuvering target. The inclusion of a maneuvering target is the element which complicates the analytic development in that the system equations become nonhomogeneous. The analytic derivation for this nonhomogeneous case is given in this paper, and the resulting optimal guidance law agrees with the intuitive result determined by Cottrell under the assumption of a target undergoing a constant acceleration.

References

[1] Cottrell, R. G., "Optimal Intercept Guidance for Short Range Tactical Missiles," *AIAA Journal*, Vol. 9, No. 7, July 1971, pp. 1414–1415.

[2] Cottrell, R. G., "Optimal and Sub-Optimal Guidance Laws for Air-to-Air Interception," Interdepartmental Note, 2342.0520/114, Nov. 13, 1969, Hughes Aircraft Co., Culver City, Calif.

[3] Bryson, A. E. and Ho, Y. C., *Applied Optimal Control*, Blaisdell, Waltham, Mass., 1968.

Correlation of the Over-All Thermal Resistance of Metallic 0-Rings Contacting Two Cylinders

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Nomenclature

A = heat flow area $(\pi D^2/4)$
C_1 = correlation parameter, Eq. (5)
C_2 = correlation parameter, Eq. (5)
C_3 = correlation parameter, Eq. (7)
Dm = mean ring diameter
di = inside tube diameter
do = outside tube diameter
E = modulus of elasticity
F = total force on 0-ring
k = thermal conductivity
P^* = dimensionless load $[F(1-v^2)/ED_m^2]$
Q = heat flow rate
R = thermal resistance
R^* = dimensionless thermal resistance $(kD_m R)$
T = temperature
t = tube wall thickness $(2t = do - di)$
t^* = dimensionless wall thickness $(2t/do)$
v = Poisson's ratio

Subscripts
1, 2, 3 = cylinders 1 and 2, and 0-ring, respectively
o = over-all

Introduction

THIS Note considers the over-all resistance to heat transfer across a joint formed by solid or hollow metallic 0-rings in contact with smooth, flat ends of cylinders, Fig. 1. Such

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